

A PERIODICITY-BASED APPROACH ON HARMONY PERCEPTION INCLUDING NON-WESTERN SCALES

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ABSTRACT

Empirical results demonstrate that human subjects rate harmonies, e.g. major and minor triads, differently with respect to their sonority. These judgements of listeners have a strong psychophysical basis. Therefore, harmony perception often is explained by the notions of dissonance and tension, focussing on the sonority of intervals and triads, i.e. chords only.

In this paper, we introduce an approach based on the notion of periodicity, providing a unified view on the perception of chords and scales. The method is psychophysically motivated: It applies the fact, that the just noticeable difference of human pitch perception is about 1% for the musically important low frequency range and that periodicities of complex chords can be detected in the human brain, as recent results from neuro-science prove. Mathematically, periodicity can be derived from the frequency ratios of the tones in the harmony with respect to its lowest tone, employing e.g. the so-called Stern-Brocot tree.

The theoretical results presented here correlate very well to experimental results on the perception of common triads. Applying the periodicity-based approach to harmonies consisting of more than three tones also yields interesting and plausible results: Standard chords in jazz and also the diatonic and blues scales have low, i.e. good harmonicity values according to the approach presented here, ranking among the top 5% in their respective tone multiplicity category. Furthermore, according to their average harmonicity value, all church modes rank among the top 11 of 462 theoretically possible scales with 7 tones. Therefore, the periodicity-based approach can contribute to the discussion about the origin of chords and scales, especially in western music. It adopts equal temperament as reference system.

Non-western scales, e.g. in Turkish classical music with makam melody types or octave reduced tone scales of recordings of traditional Central African music, do not seem to be based on equal temperament tunings. Nevertheless, they can be successfully analyzed by the periodicity-based approach, predicting also relatively good values of sonority for these scales. Thus in summary it seems that both the perception of chords and scales in any music obey similar principles, namely that simpler patterns with short period lengths are preferred by humans. They correspond to higher sonority.

1. INTRODUCTION

1.1 Motivation

Music perception and composition seem to be influenced not only by convention or culture, manifested by musical styles or composers,

but also by the psychophysics of tone perception (Langner, 2007; Parncutt, 1989; Roederer, 2008). Thus, in order to better understand the process of musical creativity, the following questions should be addressed: 1. What are underlying (psychophysical) principles of music perception? 2. How can the perceived sonority of chords and scales be explained?

Therefore, in the rest of this section (Sect. 1), we will introduce basic musical notions and results. After that, we will briefly review existing psychophysical theories on harmony perception (Sect. 2), which are often based on the notions dissonance and tension, taking harmonic overtone spectra into account. In contrast to this, the approach presented here (Sect. 3) is simply based on the periodicity of chords. Applying this theory to common musical chords and also scales, shows a very good correlation to empirical results, that e.g. most subjects prefer major to minor chords (Sect. 4). Furthermore, we will apply the periodicity-based approach also to non-western music, dropping the assumption that music is based on some equal temperament tone scale. Finally, we will end up with conclusions (Sect. 5).

1.2 Basic Musical Notions

Before we are able to address the problem of harmony perception, we should clarify the terminology we use. For this, we follow the lines of Parncutt (1989). The basic entity we have to deal with is a *tone*: A *pure* tone is a tone with a sinusoidal waveform. It has a specific pitch, corresponding to its perceived *frequency* f , usually measured in Hertz (Hz), i.e. periods per second. In practice, pure tones almost never appear. The tones produced by real instruments like strings, tubes, or the human voice have harmonic or other *overtones*. The frequencies of harmonic overtones are integer multiples of a fundamental frequency f . For the frequency of the n -th overtone ($n \geq 1$), it holds $f_n = n \cdot f$, i.e. $f_1 = f$. The amplitudes of the overtones define the *spectrum* of a tone or sound and account for its loudness and specific timbre.

A *harmony* in an abstract sense can be identified by a set of tones forming an interval, chord, or scale. Two tones define an *interval*, which is the distance between two pitch categories. The most prominent interval is the *octave*, corresponding to a frequency ratio of 2/1. Since the same names are assigned to notes an octave apart, they are assumed to be *octave equivalent*. An octave is usually divided into 12 semitones in western music, corresponding to a frequency ratio of $\sqrt[12]{2}$ in equal temperament (cf. Sect. 3.3). Thus, intervals may also be defined by the number of semitones between two tones. A *chord* is a complex musical sound comprising three or more simultaneous tones, while a *scale* is a set of musical notes, whose corresponding tones usually sound consecutively. Both can be identified by the numbers of semitones in the harmony.

A *triad* is a chord consisting of three tones. Classical triads are built from major and minor thirds, i.e., the distance between successive pairs of tones are 3 or 4 semitones. For example, the major triad consists of the semitones {0, 4, 7}, which is the *root position* of this chord. An *inversion* of a chord is obtained by transposing the currently lowest tone by an octave. Fig. 1 (a) shows the three inversions of the E major chord, including the root position. Fig. 1 (b)–(e) shows all triads that can be built from thirds including their inversion, always with e' as lowest tone. Fig. 1 (f) shows the suspended chord, built from perfect fourths (5 semitones). Its last inversion, consisting of the semitones {0, 5, 10}, reveals this.

2. THEORIES ON HARMONY PERCEPTION

Chord classes lead to different musical modes. The major chord is often associated with emotional terms like *happy*, *strong*, or *bright*, and, in contrast to this, the minor chord with terms like *sad*, *weak*, or *dark*. Empirical results (see e.g. Roberts, 1986) reveal a preference ordering on the perceived sonority of the triads as follows: major < minor < diminished < augmented. Since all these triads are built from thirds, thirds do not provide an explanation of this preference ordering on its own. Therefore, let us now review existing theories on harmony perception, discussing some of their merits and drawbacks.

2.1 Explanation by Overtones

Overtones can explain the origin of the major triad and hence its high perceived sonority. The major triad appears early in the sequence, namely overtones 4, 5, 6 (root position) and—even earlier—3, 4, 5 (second inversion). But it is well-known, that overtones fail to explain the origin of the minor chord.

2.2 Dissonance and Tension

Since the origin of harmony and scales cannot be explained well by overtones, newer explanations base upon the notions of dissonance (Terhardt et al., 1982; Parncutt, 1989; Sethares, 2005) and tension (Cook and Fujisawa, 2006). In general, *dissonance* is the opposite to consonance, meaning how well tones sound together. If two sine waves sound together, typical perceptions include pleasant beating (when the frequency difference is small), so-called roughness (when the difference grows larger), and separation into two tones (when the frequency difference increases even further). Based on these observations, several mathematical functions for dissonance curves are proposed in the literature.

Although this approach correlates better to the empirical results on harmony perception, it does not explain the low perceived sonority of the diminished or the augmented triad, which are built from two minor or major thirds, respectively. Therefore, Cook and Fujisawa (2006) adopt the argument from psychology that neighboring intervals of equivalent size are unstable and produce a sense of tonal tension, that is resolved by pitch changes leading to unequal intervals.

3. A PERIODICITY-BASED THEORY

The approaches discussed so far more or less take the frequency spectrum of a sound as their starting point. Obviously, analyzing the frequency spectrum is closely related to analyzing the time domain

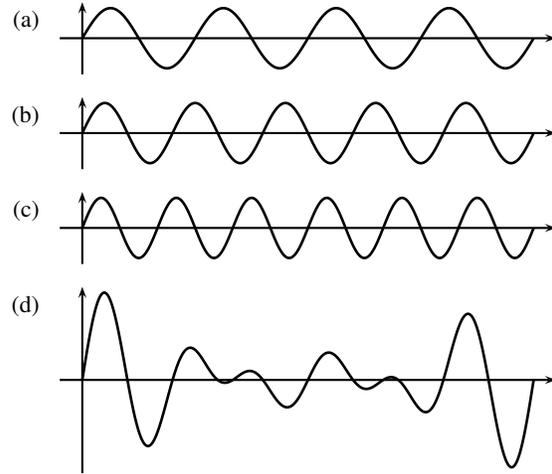


Figure 2. Sinusoids of the major triad.

(periodicity). However, subjective pitch detection, i.e., the capability of our auditory system to identify the repetition rate (periodicity) of a complex tone sensation, only works for the lower but musically important frequency range up to about 1.500 Hz (Roederer, 2008). In consequence, a *missing fundamental* tone can be assigned to each interval. The tone with the respective frequency, called *virtual pitch* of the interval, is not present as an original tone component. It has nothing to do with (first-order) beats and is perceived not directly in the ear, but in the brain.

3.1 Periodicity Pitch of Chords

For intervals, i.e. two tones, the concept of virtual pitch has been studied many times in the literature (cf. Roederer, 2008, and references therein). The idea in this paper now is to transfer this concept to chords by considering *relative periodicity*, i.e. the period length of complex sinusoids relative to the period length of the frequency of the lowest tone component (cf. Beament, 2001, Sect. 7.1). For example, the A major triad in just intonation consists of three tones with (absolute) frequencies $f_1 = 440$ Hz, $f_2 = 550$ Hz, and $f_3 = 660$ Hz. The respective frequency ratios wrt. the lowest tone (a') are $F_1 = 1/1$, $F_2 = 5/4$ (third), and $F_3 = 3/2$ (fifth), corresponding to the semitones {0, 4, 7}. Fig. 2 (a)–(c) show the sinusoids for the three pure tone components and Fig. 2 (d) their superposition, i.e. the graph of the function $\sin(\omega_1 t) + \sin(\omega_2 t) + \sin(\omega_3 t)$, where $\omega_i = 2\pi f_i$ are the respective angular frequencies, and t is the time.

As one can see, the period length of the chord is (only) four times the period length of the lowest tone for this example. In the following, we call this ratio h . It depends on the frequency ratios $\{a_1/b_1, \dots, a_k/b_k\}$ of the given chord. We assume, that each frequency ratio F_i is a fraction a_i/b_i (in its lowest terms), because otherwise no finite period length can be found in general, and it holds $F_i \approx f_i/f_1$ for $1 \leq i \leq k$. This means, all frequencies are relativized to the lowest frequency f_1 , and $F_1 = 1$. The value of h then can be computed as $\text{lcm}(b_1, \dots, b_k)$, i.e., it is the *least common multiple* (lcm) of the denominators of the frequency ratios.

3.2 A Hypothesis on Harmony Perception

We now set up the following hypothesis on harmony perception: The

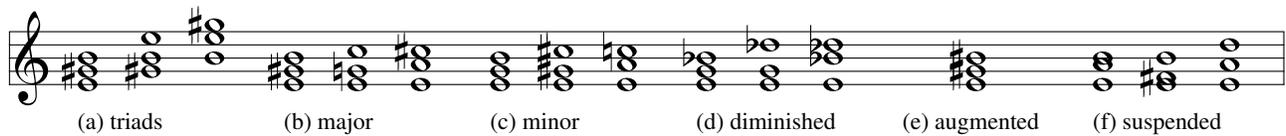


Figure 1. Triads and their inversions.

interval	k	(a) equal temperament	(b) rational
prime, unison	0	1.000	1/1 (0.00%)
minor second	1	1.059	16/15 (0.68%)
major second	2	1.122	9/8 (0.23%)
minor third	3	1.189	6/5 (0.91%)
major third	4	1.260	5/4 (-0.79%)
perfect fourth	5	1.335	4/3 (-0.11%)
tritone	6	1.414	17/12 (0.17%)
perfect fifth	7	1.498	3/2 (0.11%)
minor sixth	8	1.587	8/5 (0.79%)
major sixth	9	1.682	5/3 (-0.90%)
minor seventh	10	1.782	16/9 (-0.23%)
major seventh	11	1.888	15/8 (-0.68%)
octave	12	2.000	2/1 (0.00%)

Table 1. Table of relative frequencies for different tunings.

perceived sonority of a chord, called *harmonicity* in this context, decreases with the value of h . For the major triad in root position we have $h = 4$ (see above), which is quite low. Therefore, its predicted sonority is high. This correlates well to the empirical results, in general better than the approaches discussed in the previous section (Sect. 2). In addition, the periodicity-based theory presented here is computationally simple, because it needs no assumptions on parameters, such as harmonic overtone spectra. Neither complex summation nor computing local extrema is required. Only the frequency ratios of the tone components in the chord are needed as input parameters. But we still have to answer the question, which frequency ratios should be used in the computation of h . Since this is done in a special way here, we present this now in more detail.

3.3 Tuning and Frequency Ratios

The frequencies for the k -th semitone in *equal temperament* with twelve tones per octave can be computed as $f_k = \sqrt[12]{2}^k \cdot f_1$, where f_1 is the frequency of the lowest tone. The respective frequency ratios are shown in Tab. 1 (a). The values grow exponentially and not linearly, following the *Weber-Fechner law* in psychophysics, which says that, if the physical magnitude of stimuli grows exponentially, then the perceived intensity grows only linearly. In equal temperament, all keys sound equal. This is essential for playing in different keys on one instrument and for modulation, i.e. changing from one key to another within one piece of music. Since this seems to be universal, at least in western music, we will adopt the equal temperament as *reference system* for other tunings.

The frequency ratios in equal temperament are irrational numbers (except for the ground tone and its octaves), but for periodicity detection they must be fractions, as mentioned above. Let us thus consider other tunings with rational frequency ratios. The oldest tuning with this property is probably the *Pythagorean tuning*.

However, the Pythagorean tuning does not follow results of psy-

chophysics, e.g., that human subjects can distinguish frequency differences for pure tone components only up to a certain resolution, namely 0.5% under optimal conditions. For the musically important low frequency range, especially the tones in (accompanying) chords, this so-called *just noticeable difference* is worse, namely only below about 1% (Roederer, 2008). Therefore, we should look for tunings, where the relative error is approximately 1%. We will use the frequency ratios of the *rational tuning*, shown in Tab. 1 (b), in our analyses of harmonicity. They are fractions with smallest possible denominator, such that the relative error wrt. equal temperament is just below 1%. They can be computed by means of *Farey sequences*, i.e. ordered sequences of completely reduced fractions which have denominators less than or equal to some (small) n , or by continued fraction expansion (see Olds, 1963).

3.4 Approximating Numbers by Fractions

Thus, we must be able to approximate frequency ratios $\{F_1, \dots, F_k\}$ by fractions. They should all have the same, smallest possible denominator n . It corresponds to the overall period length of the harmony, i.e. its harmonicity. Clearly, the best approximation of the frequency ratio $x = F_i$ by a fraction with denominator n is $y = [n F_i]/n$. Here, $[\cdot]$ is the rounding function, that rounds its argument to the nearest integer number. Now, n can be determined as the smallest integer number such that the relative error $|y/x - 1|$ is below some given threshold —1% in our case— for all F_i (with $1 \leq i \leq k$).

This straightforward procedure obviously is applicable to a set of frequency ratios. Nevertheless, for intervals only one value $F > 1$ has to be considered. For this, the so-called Stern-Brocot tree can be employed (cf. Graham et al., 1994; Forišek, 2007). It induces an effective algorithm for approximating rational numbers x by fractions a/b with some precision (i.e. maximal relative error). The main idea is to perform some binary search (see Fig. 3) between two bounds: a_l/b_l (left) and a_r/b_r (right). We start with the two integer numbers that are nearest to x , and repeat computing the so-called *mediant* $a_m/b_m = (a_l + a_r)/(b_l + b_r)$, until x is approximated with the desired precision p . Fig. 3 shows an improved, efficient version of this procedure, following the lines of Forišek (2007).

4. APPLICATION OF THE THEORY

Let us now apply the periodicity-based theory to common musical chords and correlate the obtained results with empirical results. Tab. 2 shows the perceived and computed relative sonority of basic chord classes (cf. Fig. 1). Tab. 2 (a) shows the ranking for the perceived sonority according to empirical experiments reported in Roberts (1986), which have been repeated by many others with similar results. Unfortunately, Roberts (1986) does not consider the suspended triad. Therefore, it is not ranked in the table.

Tab. 2 (b) shows the results of some other experiment (Cook, 2009). Here, subjects have been asked to evaluate isolated grand piano (midi) triads wrt. their perceived *brightness* versus *darkness* on a

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 $x_{\min} = (1 - p)x;$ 
 $x_{\max} = (1 + p)x;$ 
 $a_l/b_l = \lfloor x \rfloor / 1;$ 
 $a_r/b_r = (\lfloor x \rfloor + 1) / 1;$ 
loop
   $a_m/b_m = (a_l + a_r) / (b_l + b_r);$ 
  if  $x_{\min} \leq a_m/b_m \leq x_{\max}$  then
    return  $a_m/b_m;$ 
  else if  $x_{\max} < a_m/b_m$  then
     $k = \lfloor (a_r - x_{\max} b_r) / (x_{\max} b_l - a_l) \rfloor;$ 
     $a_m/b_m = (a_r + k a_l) / (b_r + k b_l);$ 
     $a_r/b_r = a_m/b_m$ 
  else if  $a_m/b_m < x_{\min}$  then
     $k = \lfloor (x_{\min} b_l - a_l) / (a_r - x_{\min} b_r) \rfloor;$ 
     $a_m/b_m = (a_l + k a_r) / (b_l + k b_r);$ 
     $a_l/b_l = a_m/b_m$ 
  end if
end loop

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Figure 3. Approximating rational numbers x by fractions a/b with precision p . We use the floor function $\lfloor x \rfloor$ here, which yields the largest integer less than or equal to x .

scale of 1-5 among others. Unfortunately, the inversions of a triad are not distinguished in this experiment. As one can see, the suspended chord is ranked lower than major and minor, but the difference between the minor and the suspended chords is smaller than the one between the suspended and the diminished chord wrt. the scalar values (shown in brackets).

4.1 Degree of Softness

The so-called *gradus suavitatis* (degree of softness) (Euler, 1739; Bailhache, 1997) is mathematically related to harmonicity as introduced here, but has some important differences: Euler (1739) considers the least common multiple (lcm) of denominators and numerators of the involved frequency ratios $\{a_1/b_1, \dots, a_k/b_k\}$, whereas the harmonicity notion used here refers to the denominators only.

Let now $p_1^{m_1} \cdot \dots \cdot p_n^{m_n}$ be the prime factorization of $\text{lcm}(a_1, \dots, a_k, b_1, \dots, b_k)$. Then, the degree of softness is defined by $1 + \sum_{i=1}^n m_i \cdot (p_i - 1)$, while we do some averaging and apply the current knowledge on periodicity detection in the brain and on the just noticeable difference in pitch perception (see Sect. 3.3 and Sect. 4.2). Tab. 2 (c) shows the ranking of triads according to Euler (1739) wrt. the rational tuning. The gradus values are given in brackets. Obviously, the order does not reflect the empirically determined ranking of triads, whereas it does with harmonicity, as we will see next.

4.2 Comparison with Harmonicity

Tab. 2 (d)–(e) shows the ranking wrt. the harmonicity values h . As one can see, there is almost a one-to-one correspondence with the empirical results. The numbers in brackets are the respective harmonicity values h and h^* , where the latter are averaged over all inversions. For details of this procedure, the reader is referred to Stolzenburg (2009).

Tab. 2 (a) and (e) differ only in two respects: First, the most consonant chord according to harmonicity (rank 1) is the second inversion of the major triad with semitones $\{0, 5, 9\}$ and not the root position. Its calculated harmonicity is $h = 3$, which however coin-

cides with the fact, that the second inversion appears before the root position in the harmonic overtone sequence (cf. Sect. 2.1). Second, the augmented triad appears late as expected (rank 11 of 13), but the root position and the second inversion of the diminished triad appear still later. However, the continued fraction expansion for the tritone (semitone 6, frequency ratio $\sqrt{2}$), occurring in both triads, yields first $7/5$, which is only slightly mistuned. This would lead to a significantly lower h value of the two chords – as desired. Thus, in summary, the periodicity-based approach on harmony perception fits very well to empirical results. Nevertheless, it does not explain the slightly lower perceived sonority of the suspended chord (cf. Cook, 2009).

4.3 From Chords to Scales

The periodicity-based approach can easily be applied to scales and yields still meaningful results (see Stolzenburg, 2009). For instance, according to their h^* value, all church modes (Ionian, Dorian, Phrygian, Lydian, Mixolydian, Aeolian, Locrian) i.e. the diatonic scale and its inversions, rank among the top 11 of 462 possible scales with 7 tones. Therefore, the periodicity-based theory can contribute significantly to the discussion about the origin of scales of western music.

Furthermore, the approach can be adapted also to non-western music. For this, the implicit assumption that music is always based on some equal temperament cannot be kept up any longer. Nonetheless, harmonicity of scales can be computed without reference to any tuning. Applying the straightforward procedure from Sect. 3.4 to the western church modes, yields periodicity values n between 24.3 and 27.8 (after averaging). Interestingly, the values for scales of makam melody types in Turkish classical music and for some octave reduced tone scales of recordings of traditional Central African music (see <http://music.africamuseum.be/> and Moelants et al., 2009) that consist of 7 tones are approximately in the same range. The 6 mainly used scales in Turkish classical music (Rast, Nihavent, Uşşak, Hüseyini, Hicaz, Hüzam) have average periodicity values between 27.6 and 30.4. This point should be further investigated.

5. CONCLUSIONS

We have seen in this paper, that harmony perception can be explained well by considering relative periodicities of chords, that can be computed from the frequency ratios of the intervals in the chord. The approach shows a good correlation to empirical studies on perceived sonority. Even the origin of scales can be described with this approach. It is mathematically simple, employing Farey sequences or the Stern-Brocot tree, respectively. The approach has a strong psychophysical basis. It takes into account that human pitch perception is limited by a just noticeable difference of about 1% and assumes that virtual pitch of chords (chord periodicity) can be detected. The latter is indeed possible, as results from neuro-science prove (Bahmer and Langner, 2006; Cariani, 1999; Langner, 2007; Meddis and Hewitt, 1991).

From the good correlation of the periodicity-based theory with the empirical results presented here, one may conclude, that there is a strong psychophysical basis for harmony perception and the origin of musical scales. As underlying principle for this, periodicity detection turns out to be more important than spectral analysis, although cultural and other aspects certainly must not be neglected. The question, how different harmonies cause different emotional effects like happiness or sadness is not yet answered by this, of course. For this,

chord class		(a) Roberts (1986)	(b) Cook (2009)	(c) Euler (1739)	(d) harmonicity	(e) harmonicity*
major	{0, 4, 7}	1	1	(4.07)	6-9 (9)	2 (4)
	{0, 3, 8}	2			10-11 (10)	3 (5)
	{0, 5, 9}	3			6-9 (9)	1 (3)
minor	{0, 3, 7}	4	2	(3.09)	2-5 (8)	7 (10)
	{0, 4, 9}	5			6-9 (9)	8-9 (12)
	{0, 5, 8}	6			10-11 (10)	10-11 (15)
suspended	{0, 5, 7}		3	(2.91)	1 (5)	4 (6)
	{0, 2, 7}				2-5 (8)	5 (8)
	{0, 5, 10}				6-9 (9)	6 (9)
diminished	{0, 3, 6}	7	4	(2.43)	12-13 (25)	13 (60)
	{0, 3, 9}	8			2-5 (8)	10-11 (15)
	{0, 6, 9}	9			12-13 (25)	8-9 (12)
augmented	{0, 4, 8}	10	5	(2.16)	2-5 (8)	12 (20)
						11 (19.7)

Table 2. Ranking relative sonorities of common triads.

the working of the brain must be better understood. In consequence, models of the brain that take temporal properties into account should be investigated (claimed also in Roederer, 2008). For this, (artificial) neural networks that have this property (Cariani, 2001; Haykin, 2008; Stolzenburg and Ruh, 2009) should be considered further.

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